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# Strong and $\triangle$-convergence of some iterative schemes in CAT(0) spaces

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ABSTRACT

In this paper, we get some results on strong and $\triangle$-convergence in CAT(0) spaces for an iterative scheme which is both faster than and independent of the Ishikawa scheme. We also obtain some results for two mappings using the Ishikawa-type iteration scheme. The motivation of the present work comes from that of Dhompongsa and Panyanak (2008) [3].



## 1. Introduction and preliminaries

The concept of $\triangle$-convergence in general metric spaces was coined by Lim [1]. Kirk and Panyanak [2] specialized this concept to CAT(0) spaces and showed that many Banach space results involving weak convergence have precise analogs in this setting. Dhompongsa and Panyanak [3] continued to work in this direction. Their results involved the Mann and Ishikawa iteration schemes involving one mapping. In this paper, we approximate common fixed points of two nonexpansive mappings by an iteration scheme which is both independent and simpler than the Ishikawa-type iteration scheme.

Let us recall some basics. A metric space $X$ is called a CAT(0) space [4] if it is geodesically connected and if every geodesic triangle in $X$ is at least as "thin" as its comparison triangle in Euclidean plane. For a vigorous discussion, see Bridson and Haefliger [5] or Burago–Burago–Ivanov [6]. The complex Hilbert ball with a hyperbolic metric is a CAT(0) space; see [7,8].

Let $(X, d)$ be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from $x$ to $y$) is a map $c$ from a closed interval $[0, l] \subset \mathbb{R}$ to $X$ such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, $c$ is an isometry and $d(x, y) = l$. The image of $c$ is called a geodesic (or metric) segment joining $x$ and $y$. When it is unique this geodesic segment is denoted by $[x, y]$. The space $(X, d)$ is said to be a geodesic space if every two points of $X$ are joined by a geodesic, and $X$ is said to be uniquely geodesic if there is exactly one geodesic joining $x$ and $y$ for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if $Y$ includes every geodesic segment joining any two of its points. A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space $(X, d)$ consists of three points $x_1, x_2, x_3$ in $X$ (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in $(X, d)$ is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. A geodesic space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

CAT(0) : Let $\triangle$ be a geodesic triangle in $X$ and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \overline{\triangle}$,

$$d(x, y) \le d_{\mathbb{E}^2}(\bar{x}, \bar{y}).$$

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If $x, y_1, y_2$ are points in a CAT(0) space and if $y_0$ is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2. \tag{CN}$$

This is the (CN) inequality of Bruhat and Tits [9]. In fact (cf. [5], p. 163), a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality.

Following are some elementary facts about CAT(0) spaces; cf. [3].

**Lemma 1.** *Let $(X, d)$ be a CAT(0) space. Then*

(i) *$(X, d)$ is uniquely geodesic.*
(ii) *Let $p, x, y$ be points of $X$, let $\alpha \in [0, 1]$, and let $m_1$ and $m_2$ denote, respectively, the points of $[p, x]$ and $[p, y]$ satisfying $d(p, m_1) = \alpha d(p, x)$ and $d(p, m_2) = \alpha d(p, y)$. Then*

$$d(m_1, m_2) \leq \alpha d(x, y). \tag{1.1}$$

(iii) *Let $x, y \in X, x \neq y$ and $z, w \in [x, y]$ such that $d(x, z) = d(x, w)$. Then $z = w$.*
(iv) *Let $x, y \in X$. For each $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that*

$$d(x, z) = td(x, y) \quad \text{and} \quad d(y, z) = (1 - t)d(x, y). \tag{1.2}$$

For convenience, from now on we will use the notation $(1 - t)x \oplus ty$ for the unique point $z$ satisfying (1.2).

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space $X$. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known (see, e.g., [10], Proposition 7) that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\}$ in $X$ is said to $\triangle$-converge to $x \in X$ if $x$ is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\triangle\text{-}\lim_n x_n = x$ and call $x$ the $\triangle$-limit of $\{x_n\}$; see [2,1]. We denote $w_\triangle(x_n) := \bigcup\{A(\{u_n\})\}$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$.

The following lemmas can be found in [3].

**Lemma 2** (*[3], Lemma 2.4*)**.** *Let $X$ be a CAT(0) space. Then $d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z)$ for all $x, y, z \in X$ and $t \in [0, 1]$.*

**Lemma 3** (*[3], Lemma 2.5*)**.** *Let $X$ be a CAT(0) space. Then*

$$d((1 - t)x \oplus ty, z)^2 \leq (1 - t)d(x, z)^2 + td(y, z)^2 - t(1 - t)d(x, y)^2$$

*for all $x, y, z \in X$ and $t \in [0, 1]$.*

**Lemma 4** (*[3], Lemma 2.7*)**.**

(i) *Every bounded sequence in $X$ has a $\triangle$-convergent subsequence.*
(ii) *If $C$ is a closed convex subset of $X$ and if $\{x_n\}$ is a bounded sequence in $C$, then the asymptotic center of $\{x_n\}$ is in $C$.*
(iii) *If $C$ is a closed convex subset of $X$ and if $f : C \to X$ is a nonexpansive mapping, then the conditions, $\{x_n\}$ $\triangle$-converges to $x$ and $d(x_n, f(x_n)) \to 0$, imply $x \in C$ and $f(x) = x$.*

The Picard iterative process is defined by the sequence $\{x_n\}$:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = Tx_n, \quad n \in \mathbb{N}. \end{cases} \tag{1.3}$$

The Mann iterative process is defined by the sequence $\{x_n\}$:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1 - a_n)x_n + a_nTx_n, \quad n \in \mathbb{N} \end{cases} \tag{1.4}$$

where $\{a_n\}$ is in $(0, 1)$.

The sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1 - a_n)x_n + a_n T y_n, \\ y_n = (1 - b_n)x_n + b_n T x_n, \quad n \in \mathbb{N} \end{cases} \tag{1.5}$$

where $\{a_n\}$ and $\{b_n\}$ are in $(0, 1)$, is known as the Ishikawa iterative process.

Recently, Agarwal et al. [11] introduced the following iterative process:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1 - a_n)T x_n + a_n T y_n, \\ y_n = (1 - b_n)x_n + b_n T x_n, \quad n \in \mathbb{N} \end{cases} \tag{1.6}$$

where $\{a_n\}$ and $\{b_n\}$ in $(0, 1)$.

Note that (1.6) is independent of (1.5) (and hence of (1.4)). Agarwal et al. [11] showed (see Proposition 3.1) that (1.6) converges at a rate same as that of Picard iteration and faster than Mann iteration for contractions and it is not hard to see on similar lines that scheme (1.6) also converges faster than the Ishikawa iteration scheme.

Dhompongsa and Panyanak [3] studied the $\triangle$-convergence of the Picard, Mann and Ishikawa iterates (Theorems 3.1, 3.2 and 3.3 respectively in [3]). While acknowledging their contribution, we note that their schemes involve one mapping. The case of two mappings in iteration processes has also remained under study since Das and Debata [12] gave and studied a two mappings' scheme on the pattern of the Ishikawa scheme:

$$\begin{cases} x_{n+1} = (1 - a_n)x_n + a_n T y_n, \\ y_n = (1 - b_n)x_n + b_n S x_n, \quad n \in \mathbb{N}. \end{cases} \tag{1.7}$$

Also see, for example, [13,14]. This scheme reduces to the Ishikawa scheme when $S = T$ and to Mann iteration scheme when $S = I$. Note that two mappings' case, that is, approximating the common fixed points, has its own importance as it has a direct link with the minimization problem; see for example [15].

We now modify (1.6) and (1.7) in CAT(0) spaces as follows.

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1 - a_n)T x_n \oplus a_n T y_n, \\ y_n = (1 - b_n)x_n \oplus b_n T x_n, \quad n \in \mathbb{N} \end{cases} \tag{1.8}$$

and

$$\begin{cases} x_{n+1} = (1 - a_n)x_n \oplus a_n T y_n, \\ y_n = (1 - b_n)x_n \oplus b_n S x_n, \quad n \in \mathbb{N} \end{cases} \tag{1.9}$$

where $\{a_n\}$ and $\{b_n\}$ are in $(0, 1)$.

Our purpose in this paper is twofold.

(i) To get some results on strong and $\triangle$-convergence in CAT(0) spaces for (1.8). These results are independent of those proved for (1.5) (and hence for (1.4)).
(ii) To get some results for two mappings using (1.9). These results contain the results proved for (1.5) (and hence for (1.4)).

## 2. Main results

### 2.1. One mapping case

Now we are all set to prove our main results. In what follows, $F(T)$ denotes the set of fixed points of $T$. We start with proving a key lemma for later use.

**Lemma 5.** *Let $C$ be a nonempty closed convex subset of $X$. Let $T$ be a nonexpansive mapping of $C$. Let $\{a_n\}, \{b_n\}$ be such that $0 < a \le a_n, b_n \le b < 1$ for all $n \in \mathbb{N}$ and for some $a, b$. Let $\{x_n\}$ be defined by the iteration process (1.8). Then*

(i) *$\lim_{n\to\infty} d(x_n, q)$ exists for all $q \in F(T)$.*
(ii) *$\lim_{n\to\infty} d(x_n, T x_n) = 0$.*

**Proof.** Let $q \in F(T)$. Then by Lemma 2,

$$\begin{aligned} d(x_{n+1}, q) &= d((1 - a_n)T x_n \oplus a_n T y_n, q) \\ &\le (1 - a_n)d(T x_n, q) + a_n d(T y_n, q) \\ &\le (1 - a_n)d(x_n, q) + a_n d(y_n, q). \end{aligned} \tag{2.1}$$

But

$$\begin{aligned} d(y_n, q) &= d((1-b_n)x_n \oplus b_n Tx_n, q) \\ &\le (1-b_n)d(x_n, q) + b_n d(Tx_n, q) \\ &\le d(x_n, q). \end{aligned} \tag{2.2}$$

Combining (2.1) and (2.2), we have

$$d(x_{n+1}, q) \le d(x_n, q). \tag{2.3}$$

This shows that $\{d(x_n, q)\}$ is decreasing and this proves part (i). Let

$$\lim_{n\to\infty} d(x_n, q) = c. \tag{2.4}$$

To prove part (ii), we first prove that $\lim_{n\to\infty} d(y_n, q) = c$.
By (2.1),

$$d(x_{n+1}, q) \le (1-a_n)d(x_n, q) + a_n d(y_n, q).$$

This gives that

$$a_n d(x_n, q) \le d(x_n, q) + a_n d(y_n, q) - d(x_{n+1}, q)$$

or

$$\begin{aligned} d(x_n, q) &\le d(y_n, q) + \frac{1}{a_n}[d(x_n, q) - d(x_{n+1}, q)] \\ &\le d(y_n, q) + \frac{1}{a}[d(x_n, q) - d(x_{n+1}, q)]. \end{aligned}$$

This gives

$$\liminf_{n\to\infty} d(x_n, q) \le \liminf_{n\to\infty} d(y_n, q) + \lim_{n\to\infty} \frac{1}{a}[d(x_n, q) - d(x_{n+1}, q)]$$

so that

$$c \le \liminf_{n\to\infty} d(y_n, q). \tag{2.5}$$

By (2.2) and (2.4),

$$\limsup_{n\to\infty} d(y_n, q) \le c.$$

Reading it together with (2.5), we get

$$\lim_{n\to\infty} d(y_n, q) = c. \tag{2.6}$$

Next, by Lemma 3,

$$\begin{aligned} d(y_n, q)^2 &= d((1-b_n)x_n \oplus b_n Tx_n, q)^2 \\ &\le (1-b_n)d(x_n, q)^2 + b_n d(Tx_n, q)^2 - b_n(1-b_n)d(x_n, Tx_n)^2 \\ &\le d(x_n, q)^2 - b_n(1-b_n)d(x_n, Tx_n)^2. \end{aligned}$$

Thus

$$b_n(1-b_n)d(x_n, Tx_n)^2 \le d(x_n, q)^2 - d(y_n, q)^2$$

so that

$$\begin{aligned} d(x_n, Tx_n)^2 &\le \frac{1}{b_n(1-b_n)}[d(x_n, q)^2 - d(y_n, q)^2] \\ &\le \frac{1}{a(1-b)}[d(x_n, q)^2 - d(y_n, q)^2]. \end{aligned}$$

Now using (2.4) and (2.6), $\limsup d(x_n, Tx_n) \le 0$ and hence

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \quad \square$$

Before we prove our $\triangle$-convergence theorem, we remark that Kirk has proved the existence of fixed points of nonexpansive mappings in CAT(0) spaces; see [16, Theorem 12].

**Theorem 1.** *Let $X, C, T, \{a_n\}, \{b_n\}$ and $\{x_n\}$ be as in Lemma 5, Then $\{x_n\}\triangle$-converges to a fixed point of $T$.*

**Proof.** By Lemma 5, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Also, $\lim_{n\to\infty} d(x_n, q)$ exists for all $q \in F(T)$. Thus $\{x_n\}$ is bounded. First, we show that $w_\triangle(x_n) \subseteq F(T)$. Let $u \in w_\triangle(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 4, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle\text{-}\lim_n v_n = v$ for some $v \in C$. By Lemma 4, $v \in F(T)$. By Lemma 5, $\lim_{n\to\infty} d(x_n, v)$ exists. We now claim that $u = v$. Assume on contrary, that $u \neq v$. Then, by the uniqueness of asymptotic centers, we have

$$\begin{aligned}
\limsup_{n\to\infty} d(v_n, v) &< \limsup_{n\to\infty} d(v_n, u)\\
&\leq \limsup_{n\to\infty} d(u_n, u)\\
&< \limsup_{n\to\infty} d(u_n, v)\\
&= \limsup_{n\to\infty} d(x_n, v)\\
&= \limsup_{n\to\infty} d(v_n, v),
\end{aligned}$$

a contradiction. Thus, $u = v \in F(T)$ and hence $w_\triangle(x_n) \subseteq F(T)$. To show that $\{x_n\}\triangle$-converges to a fixed point of $T$, we show that $w_\triangle(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemma 4, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle\text{-}\lim_n v_n = v$ for some $v \in C$. Let $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. We have already seen that $u = v$ and $v \in F(T)$. Finally, we claim that $x = v$. If not, then existence of $\lim_{n\to\infty} d(x_n, v)$ and uniqueness of asymptotic centers imply that

$$\begin{aligned}
\limsup_{n\to\infty} d(v_n, v) &< \limsup_{n\to\infty} d(v_n, x)\\
&\leq \limsup_{n\to\infty} d(x_n, x)\\
&< \limsup_{n\to\infty} d(x_n, v)\\
&= \limsup_{n\to\infty} d(v_n, v),
\end{aligned}$$

a contradiction and hence $x = v \in F(T)$. Therefore, $w_\triangle(x_n) = \{x\}$. □

**Theorem 2.** *Let $X$ be a complete $CAT(0)$ space and $C, T, \{a_n\}, \{b_n\}$ and $\{x_n\}$ be as in Lemma 5. Then $\{x_n\}$ converges strongly to a fixed point of $T$ if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, where $d(x, F(T)) = \inf\{d(x, p) : p \in F(T)\}$.*

**Proof.** Necessity is obvious.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. As proved in Lemma 5, we have

$$d(x_{n+1}, p) \leq d(x_n, p),$$

for all $p \in F(T)$. This implies that

$$d(x_{n+1}, F(T)) \leq d(x_n, F(T))$$

so that $\lim_{n\to\infty} d(x_n, F(T))$ exists. Thus by hypothesis $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence in $C$. Let $\epsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, there exists a positive integer $n_0$ such that

$$d(x_n, F(T)) < \frac{\epsilon}{4}, \quad \forall n \geq n_0.$$

In particular, $\inf\{d(x_{n_0}, p) : p \in F(T)\} < \frac{\epsilon}{4}$. Thus there must exist $p^* \in F(T)$ such that

$$d(x_{n_0}, p^*) < \frac{\epsilon}{2}.$$

Now, for all $m, n \geq n_0$, we have

$$\begin{aligned}
d(x_{n+m}, x_n) &\leq d(x_{n+m}, p^*) + d(p^*, x_n)\\
&\leq 2d(x_{n_0}, p^*)\\
&< 2\left(\frac{\epsilon}{2}\right) = \epsilon.
\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset $C$ of a complete $CAT(0)$ space and so it must converge to a point $q$ in $C$. Now, $\lim_{n\to\infty} d(x_n, F(T)) = 0$ gives that $d(q, F(T)) = 0$ and closedness of $F(T)$ forces $q$ to be in $F(T)$. □

Senter and Dotson [17] introduced the condition $(A)$ as follows.

A mapping $T : C \to C$ is said to satisfy the condition $(A)$ if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$ such that $d(x, Tx) \geq f(d(x, F(T)))$ for all $x \in C$. It is worth noting that, in the case of nonexpansive mappings, the condition $(A)$ is weaker than the compactness of $C$.

**Theorem 3.** *Let $X$ be a CAT(0) space, and $C, T, \{a_n\}, \{b_n\}$ and $\{x_n\}$ be as in Lemma 5. Let $T : C \to C$ satisfy the condition $(A)$. Then $\{x_n\}$ converges strongly to a fixed point of $T$.*

**Proof.** By Lemma 5, $\lim_{n\to\infty} d(x_n, x^*)$ exists for all $x^* \in F(T)$. Let this limit be $c$ where $c \geq 0$.

If $c = 0$, there is nothing to prove.

Suppose that $c > 0$. Now, $d(x_{n+1}, x^*) \leq d(x_n, x^*)$ gives

$$\inf_{x^*\in F} d(x_{n+1}, x^*) \leq \inf_{x^*\in F} d(x_n, x^*),$$

which means that $d(x_{n+1}, F(T)) \leq d(x_n, F(T))$ and so $\lim_{n\to\infty} d(x_n, F(T))$ exists. By using the condition $(A)$, $\lim_{n\to\infty} f(d(x_n, F(T))) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0$. Thus we have

$$\lim_{n\to\infty} f(d(x_n, F(T))) = 0.$$

Since $f$ is a nondecreasing function and $f(0) = 0$, it follows that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. The conclusion now follows from Theorem 2. $\square$

### 2.2. Two mappings' case

From now on $F$ denotes the set of common fixed points of $T$ and $S$.

**Lemma 6.** *Let $C$ be a nonempty closed convex subset of $X$. Let $T$ and $S$ be two nonexpansive mappings of $C$. Let $\{a_n\}, \{b_n\}$ be such that $0 < a \leq a_n, b_n \leq b < 1$ for all $n \in \mathbb{N}$ and for some $a, b$. Let $\{x_n\}$ be defined by the iteration process (1.9). Then*

(i) $\lim_{n\to\infty} d(x_n, q)$ *exists for all* $q \in F$.
(ii) $\lim_{n\to\infty} d(x_n, Tx_n) = 0 = \lim_{n\to\infty} d(x_n, Sx_n)$.

**Proof.** Let $q \in F$. Then by Lemma 2,

$$\begin{aligned} d(x_{n+1}, q) &= d((1 - a_n)x_n \oplus a_nTy_n, q) \\ &\leq (1 - a_n)d(x_n, q) + a_nd(Ty_n, q) \\ &\leq (1 - a_n)d(x_n, q) + a_nd(y_n, q). \end{aligned} \tag{2.7}$$

But

$$\begin{aligned} d(y_n, q) &= d((1 - b_n)x_n \oplus b_nSx_n, q) \\ &\leq (1 - b_n)d(x_n, q) + b_nd(Sx_n, q) \\ &\leq d(x_n, q). \end{aligned} \tag{2.8}$$

Combining (2.7) and (2.8), we have

$$d(x_{n+1}, q) \leq d(x_n, q). \tag{2.9}$$

This shows that $\{d(x_n, q)\}$ is decreasing and this proves part (i). Let

$$\lim_{n\to\infty} d(x_n, q) = c. \tag{2.10}$$

To prove part (ii), we proceed as follows.

By (2.7), we have

$$d(x_{n+1}, q) \leq (1 - a_n)d(x_n, q) + a_nd(y_n, q)$$

gives that

$$a_nd(x_n, q) \leq d(x_n, q) + a_nd(y_n, q) - d(x_{n+1}, q).$$

This gives

$$c \leq \liminf_{n\to\infty} d(y_n, q). \tag{2.11}$$

But (2.8) gives that

$$\limsup_{n\to\infty} d(y_n, q) \leq c$$

so that

$$\lim_{n\to\infty} d(y_n, q) = c. \tag{2.12}$$

Next, by Lemma 3,

$$\begin{aligned} d(y_n, q)^2 &= d((1-b_n)x_n \oplus b_n Sx_n, q)^2 \\ &\le (1-b_n)d(x_n, q)^2 + b_n d(Sx_n, q)^2 - b_n(1-b_n)d(x_n, Sx_n)^2 \\ &\le d(x_n, q)^2 - b_n(1-b_n)d(x_n, Sx_n)^2. \end{aligned}$$

Now using (2.10) and (2.12), $\limsup d(x_n, Sx_n) \le 0$ and we can conclude that

$$\lim_{n\to\infty} d(x_n, Sx_n) = 0. \tag{2.13}$$

Next, from

$$\begin{aligned} d(x_{n+1}, q)^2 &= d((1-a_n)x_n \oplus a_n Ty_n, q)^2 \\ &\le (1-a_n)d(x_n, q)^2 + a_n d(Ty_n, q)^2 - a_n(1-a_n)d(x_n, Ty_n)^2 \\ &\le d(x_n, q)^2 - a_n(1-a_n)d(x_n, Ty_n)^2, \end{aligned}$$

it follows that

$$\lim_{n\to\infty} d\,(x_n, Ty_n) = 0. \tag{2.14}$$

Now using (1.9),

$$\begin{aligned} d(y_n, x_n) &= d((1-b_n)x_n \oplus b_n Sx_n, x_n) \\ &\le (1-b_n)d(x_n, x_n) + b_n d(Sx_n, x_n). \end{aligned}$$

This implies by (2.13),

$$\lim_{n\to\infty} d(y_n, x_n) = 0. \tag{2.15}$$

Finally,

$$\begin{aligned} d(x_n, Tx_n) &\le d(x_n, Ty_n) + d(Ty_n, Tx_n) \\ &\le d(x_n, Ty_n) + d(y_n, x_n) \end{aligned}$$

yields

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0$$

as desired. □

The existence of common fixed points of nonexpansive mappings in CAT(0) spaces has already been established by Dhompongsa et al.; see [18, Theorem 4.1]. The technique of the proof of the following theorem is different from that used by Laowang and Panyanak [19, Theorem 3.6].

**Theorem 4.** *Let $X, C, T, S, \{a_n\}, \{b_n\}$ and $\{x_n\}$ be in Lemma 6, Then $\{x_n\}\triangle$-converges to a common fixed point of $T$ and $S$.*

**Proof.** Let $q \in F$. Then by Lemma 6, $\lim_{n\to\infty} d(x_n, q)$ exists for all $q \in F$. Thus $\{x_n\}$ is bounded. Also, Lemma 6, gives that $\lim_{n\to\infty} d(x_n, Tx_n) = \lim_{n\to\infty} d(x_n, Sx_n) = 0$. First, we show that $w_\triangle(x_n) \subseteq F$. Let $u \in w_\triangle(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 4, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle\text{-}\lim_n v_n = v$ for some $v \in C$. We obtain $v \in F$ by a repeated application of Lemma 4 on $T$ and $S$. By Lemma 6, $\lim_{n\to\infty} d(x_n, v)$ exists. The rest of the proof closely follows the proof of Theorem 1 and is therefore omitted. □

**Theorem 5.** *Let $X$ be a complete CAT(0) space and $C, \{x_n\}, S$ and $T$ be as in Lemma 6. If $F \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of $S$ and $T$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf\{d(x, p) : p \in F\}$.*

**Proof.** Necessity is obvious.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. As proved in Lemma 6, we have

$$d(x_{n+1}, p) \le d(x_n, p),$$

for all $p \in F$. This implies that

$$d(x_{n+1}, F) \le d(x_n, F)$$

so that $\lim_{n\to\infty} d(x_n, F)$ exists. Thus by hypothesis $\lim_{n\to\infty} d(x_n, F) = 0$. The rest of the proof follows the lines of Theorem 2. □

Following Senter and Dotson [17], Khan and Fukhar-ud-din [20] introduced the so-called condition $(A')$ for two mappings and gave an improved version of it in [21] as follows: Two mappings $S, T : C \to C$ are said to satisfy the condition $(A')$ if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that either $d(x, Tx) \geq f(d(x, F))$ for all $x \in C$ or $d(x, Sx) \geq f(d(x, F))$ for all $x \in C$. This condition reduces to the condition $(A)$ of Senter and Dotson [17].

We use the condition $(A')$ to study strong convergence of $\{x_n\}$ defined by (1.9). It is worth noting that, in the case of nonexpansive mappings $S, T : C \to C$, the condition $(A')$ is weaker than the compactness of $C$.

**Theorem 6.** *Let $X$ be a CAT(0) space, $C$ and $\{x_n\}$ be as in Lemma 6. Let $S, T : C \to C$ be two nonexpansive mappings satisfying the condition $(A')$. If $F \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of $S$ and $T$.*

**Proof.** By Lemma 6, $\lim_{n\to\infty} d(x_n, x^*)$ exists for all $x^* \in F$. Let this limit be $c$ where $c \geq 0$.

If $c = 0$, there is nothing to prove.

Suppose that $c > 0$. Now, $d(x_{n+1}, x^*) \leq d(x_n, x^*)$ gives that

$$\inf_{x^*\in F} d(x_{n+1}, x^*) \leq \inf_{x^*\in F} d(x_n, x^*),$$

which means that $d(x_{n+1}, F) \leq d(x_n, F)$ and so $\lim_{n\to\infty} d(x_n, F)$ exists. By using the condition $(A')$, either

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0$$

or

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} d(x_n, Sx_n) = 0.$$

In both the cases, we have

$$\lim_{n\to\infty} f(d(x_n, F)) = 0.$$

Since $f$ is a nondecreasing function and $f(0) = 0$, it follows that $\lim_{n\to\infty} d(x_n, F) = 0$. The rest of the proof follows the pattern of the above theorem and is therefore omitted. $\square$

**Remark 1.** Theorems 4–6 contain the corresponding theorems proved for the Ishikawa scheme when $S = T$ and for the Mann iteration scheme when $S = I$.

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